FAZODA ISSIQLIK TARQALISH TENGLAMASI UCHUN QO'YILGAN KOSHI MASALASI YECHIMINI ASIMPTOTIKASI

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Annotatsiya:
fazoda issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimining beshta hadgacha asimptotik yechim olingan.

Kalit so'zlar: Issiqlik tarqalish tenglamasi, karrali Laplas integrali.

Kirish: Issiqlik tarqalish tenglamasi uchun qo'yilgan Koshi masalasi yechimining asimptotikasi haqida umumiy kurslarda yoritilmaganligi, yechimning boshlang'ich shartni qanoatlanitirishini ko'rsatishdagi qiyinchiliklarni e'tborga olib, Karrali Laplas integrali asimptotik yoyilmasining tadbiqiga asoslangan holda yuqoridagi muommolarni yechish usuli keltiriladi.

Asosiy qism
$\mathbb{R}^2$ fazosda issiqlik tarqalish tenglamasi tenglamasi uchun qo'yilgan Koshi masalasi quyidagicha aniqlanadi [3]:

$$\begin{cases} \dfrac{\partial u(x_1,x_2,t)}{\partial t} = a^2\left(\dfrac{\partial^2 u(x_1,x_2,t)}{\partial x_1^2} + \dfrac{\partial^2 u(x_1,x_2,t)}{\partial x_2^2}\right) & (1) \\ u(x_1,x_2,t)\big|_{t=+0} = \psi(x_1,x_2) & (2) \end{cases}$$

(2) shartni qanoatlantiruvchi (1) tenglamaning yechimi ushbu

$$u(x_1,x_2,t) = \frac{1}{4a^2\pi t}\iint\limits_{\mathrm{R}^2} \psi(\xi_1,\xi_2) e^{-\frac{r^2}{4a^2 t}} d\xi_1 d\xi_2 \qquad (3)$$

$$\left(r = |(x_1,x_2)-(\xi_1,\xi_2)|\right)$$

formula bilan aniqlanadi.

Teorema: Faraz qilaylik $\psi(x_1,x_2)$ funksiya $C_0^\infty(R^2)$ sinfga tegishli bo'lsin. U holda (2) shartni qanoatlantiruvchi (1) tenglamani yechimi uchun $t \to 0$ da

$$u(x_1,x_2,t)=\psi(x_1,x_2)+a^2\left(\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}\right)t+\frac{a^4}{2}\left(\frac{\partial^4\psi}{\partial x_1^4}+2\frac{\partial^4\psi}{\partial x_1^2\partial x_2^2}+\frac{\partial^4\psi}{\partial x_1^4}\right)t^2+$$

$$+\frac{a^6}{6}\left(\frac{\partial^6\psi}{\partial x_1^6}+3\frac{\partial^6\psi}{\partial x_1^4\partial x_2^2}+3\frac{\partial^6\psi}{\partial x_1^2\partial x_2^4}+\frac{\partial^6\psi}{\partial x_2^6}\right)t^3+\frac{a^8}{24}\left(\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^8}+4\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^6\partial x_2{}^2}+\right.$$

$$\left.+6\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^4\partial x_2{}^4}+4\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^2\partial x_2{}^6}+\frac{\partial^8\psi(x_1,x_2)}{\partial x_2{}^8}\right)t^4+\frac{a^{10}}{120}\left(\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1{}^{10}}+5\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1{}^8\partial x_2{}^2}+\right.$$

$$\left.+10\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1{}^6\partial x_2{}^4}+10\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1{}^4\partial x_2{}^6}+5\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1{}^2\partial x_2{}^8}+\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_2{}^{10}}\right)t^5+O(t^6)$$

asimptotik tenglik o'rinli.

Isbot: Yuqoridagi (3) tenglikning o'ng tomonini quyidagi ko'rinishda yozib olamiz:

$$\frac{1}{4a^2\pi t}\iint_{R^2}\psi(\xi_1,\xi_2)e^{-\frac{\left((x_1-\xi_1)^2+(x_2-\xi_2)^2\right)}{4a^2t}}d\xi_1 d\xi_2 \qquad (4)$$

va bu integralda quyidagicha almashtirish olamiz:

$$\begin{cases} x_1-\xi_1=2ay_1 \\ x_2-\xi_2=2ay_2 \end{cases}$$

Bunday almashtirish natijasida (4) integral quyidagi ko'rinishga keladi:

$$\frac{1}{\pi t}\iint_{R^2}\psi(x_1-2ay_1,x_2-2ay_2)e^{-\frac{\left(y_1^2+y_2^2\right)}{t}}dy_1dy_2$$

Agar yuqoridagi integralda $t=\frac{1}{\lambda}$ belgilash olsak, u holda quyidagiga ega bo'lamiz.

$$\frac{\lambda}{\pi}\iint_{R^2}\psi(x_1-2ay_1,x_2-2ay_2)e^{-\lambda\left(y_1^2+y_2^2\right)}dy_1dy_2 \qquad (7)$$

[4] adabiyotdagi quyidagi teorema keltirilgan.

Teorema: Faraz qilaylik, f funksiya quyidagi

$$\iint_{R^2}e^{-(x_1^2+x_2^2)}\,|\,f(x_1,x_2)\,|\,dx_1dx_2<+\infty$$

shartni qanoatlantirib, koordinatalar boshining biror atrofida 2m marta uzluksiz xususiy hosilalarga ega bo'lsin. U holda $\lambda\to+\infty$ da

$$\iint_{R^2}e^{-\lambda(x_1^2+x_2^2)}f(x_1,x_2)dx_1dx_2=\lambda^{-1}\left(\sum_{i=0}^{m-1}a_i\lambda^{-1}+\frac{O(1)}{\lambda^m}\right)$$

asimptotik tenglik o'rinli.

Bunda a_i koeffitsiyent quyidagicha aniqlanadi:

$$a_i = \frac{\sum_{j=0}^{i} \frac{\partial^{(2i)} f(0,0)}{\partial x_1^{2(i-j)} \partial x_2^{2j}} C_{2i}^{2j} \Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!}$$

(7) integral uchun yuqoridagi teoremadan foydalanib quyidagiga ega bo'lamiz.

$$\frac{\lambda}{\pi}\iint_{R^2} \psi(x_1-2ay_1, x_2-2ay_2)e^{-\lambda\left(y_1^2+y_2^2\right)}dy_1dy_2 = \frac{1}{\pi}\left(\sum_{i=0}^{m-1} a_i\lambda^{-1} + \frac{O(1)}{\lambda^m}\right), \quad (\lambda \to +\infty) \qquad (8)$$

bunda a_i koeffisiyent quyidagicha aniqlanadi:

$$a_i = \frac{\sum_{j=0}^{i} \frac{\partial^{(2i)} \psi(x_1,x_2)}{\partial y_1^{2(i-j)} \partial y_2^{2j}} C_{2i}^{2j} \Gamma\left(i-j+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}{(2i)!}$$

Yuqoridagi formuladan foydalanib a_i koeffitsiyentlar uchun quyidagilarni hisoblaymiz.

$$a_0 = \pi\psi(x_1,x_2),$$

$$a_1 = \frac{1}{2}\left(\frac{\partial^2\psi(x_1,x_2)}{\partial y_1{}^2}\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) + \frac{\partial^2\psi(x_1,x_2)}{\partial y_2{}^2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right)\right) =$$
$$= a^2\pi\left(\frac{\partial^2\psi(x_1,x_2)}{\partial x_1{}^2} + \frac{\partial^2\psi(x_1,x_2)}{\partial x_2{}^2}\right)$$

$$a_2 = \frac{1}{4!}\left(\frac{\partial^4\psi(x_1,x_2)}{\partial y_1{}^4}\Gamma\left(2+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) + \frac{\partial^4\psi(x_1,x_2)}{\partial y_1{}^2\partial y_2{}^2}C_4^2\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right) + \right.$$
$$\left. + \frac{\partial^4\psi(x_1,x_2)}{\partial y_2{}^4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right)\right) = \frac{a^2\pi}{2}\left(\frac{\partial^4\psi(x_1,x_2)}{\partial x_1{}^4} + 2\frac{\partial^4\psi(x_1,x_2)}{\partial x_1{}^2\partial x_2{}^2} + \frac{\partial^4\psi(x_1,x_2)}{\partial x_2{}^4}\right)$$

$$a_3 = \frac{1}{6!}\left(\frac{\partial^6\psi(x_1,x_2)}{\partial y_1{}^6}\Gamma\left(3+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) + \frac{\partial^6\psi(x_1,x_2)}{\partial y_1{}^4\partial y_2{}^2}C_6^2\Gamma\left(2+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right) + \right.$$
$$\left. + \frac{\partial^6\psi(x_1,x_2)}{\partial y_1{}^2\partial y_2{}^4}C_6^4\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right) + \frac{\partial^6\psi(x_1,x_2)}{\partial y_2{}^6}\Gamma\left(\frac{1}{2}\right)\Gamma\left(3+\frac{1}{2}\right)\right) =$$
$$= \frac{a^6\pi}{6}\left(\frac{\partial^6\psi(x_1,x_2)}{\partial x_1{}^6} + 3\frac{\partial^6\psi(x_1,x_2)}{\partial x_1{}^4\partial x_2{}^2} + 3\frac{\partial^6\psi(x_1,x_2)}{\partial x_1{}^2\partial x_2{}^4} + \frac{\partial^6\psi(x_1,x_2)}{\partial x_2{}^6}\right),$$

$$a_4 = \frac{1}{8!}\left(\frac{\partial^8\psi(x_1,x_2)}{\partial y_1{}^8}\Gamma\left(4+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) + \frac{\partial^8\psi(x_1,x_2)}{\partial y_1{}^6\partial y_2{}^2}C_8^2\Gamma\left(3+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right) + \right.$$
$$+ \frac{\partial^8\psi(x_1,x_2)}{\partial y_1{}^4\partial y_2{}^4}C_8^4\Gamma\left(2+\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right) + \frac{\partial^8\psi(x_1,x_2)}{\partial y_1{}^2\partial y_2{}^6}C_8^6\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(3+\frac{1}{2}\right) +$$

$$\left. + \frac{\partial^8\psi(x_1,x_2)}{\partial y_2{}^8}\Gamma\left(\frac{1}{2}\right)\Gamma\left(4+\frac{1}{2}\right)\right) = \frac{a^8\pi}{24}\left(\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^8} + 4\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^6\partial x_2{}^2} + \right.$$
$$\left. + 6\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^4\partial x_2{}^4} + 4\frac{\partial^8\psi(x_1,x_2)}{\partial x_1{}^2\partial x_2{}^6} + \frac{\partial^8\psi(x_1,x_2)}{\partial x_2{}^8}\right)$$

$$a_5=\frac{1}{10!}\left(\frac{\partial^{10}\psi(x_1,x_2)}{\partial y_1^{10}}\Gamma\left(5+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)+\frac{\partial^{10}\psi(x_1,x_2)}{\partial y_1^{8}\partial y_2^{2}}C_{10}^{2}\Gamma\left(4+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right)+\right.$$

$$+\frac{\partial^{10}\psi(x_1,x_2)}{\partial y_1^{6}\partial y_2^{4}}C_{10}^{4}\Gamma\left(3+\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right)+\frac{\partial^{10}\psi(x_1,x_2)}{\partial y_1^{4}\partial y_2^{6}}C_{10}^{6}\Gamma\left(2+\frac{1}{2}\right)\Gamma\left(3+\frac{1}{2}\right)+$$

$$\left.+\frac{\partial^{10}\psi(x_1,x_2)}{\partial y_1^{2}\partial y_2^{8}}C_{10}^{8}\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(4+\frac{1}{2}\right)+\frac{\partial^{10}\psi(x_1,x_2)}{\partial y_2^{10}}\right)+\frac{a^{10}\pi}{120}\left(\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{10}}+\right.$$

$$\left.+5\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{8}\partial x_2^{2}}+10\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{6}\partial x_2^{4}}+10\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{4}\partial x_2^{6}}+5\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{2}\partial x_2^{8}}+\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_2^{10}}\right)$$

Yuqorida topilgan koeffitsiyentlarni (8) yoyilmaga olib borib qo'ysak va
$t=\frac{1}{\lambda}$ ekanligidan (3) tenglik qo'yidagi ko'rinishga keladi.

$$u(x_1,x_2,t)=\psi(x_1,x_2)+a^2\left(\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}\right)t+\frac{a^4}{2}\left(\frac{\partial^4\psi}{\partial x_1^4}+2\frac{\partial^4\psi}{\partial x_1^2\partial x_2^2}+\frac{\partial^4\psi}{\partial x_1^4}\right)t^2+$$

$$+\frac{a^6}{6}\left(\frac{\partial^6\psi}{\partial x_1^6}+3\frac{\partial^6\psi}{\partial x_1^4\partial x_2^2}+3\frac{\partial^6\psi}{\partial x_1^2\partial x_2^4}+\frac{\partial^6\psi}{\partial x_2^6}\right)t^3+\frac{a^8}{24}\left(\frac{\partial^8\psi(x_1,x_2)}{\partial x_1^8}+4\frac{\partial^8\psi(x_1,x_2)}{\partial x_1^6\partial x_2^2}+\right.$$

$$\left.+6\frac{\partial^8\psi(x_1,x_2)}{\partial x_1^4\partial x_2^4}+4\frac{\partial^8\psi(x_1,x_2)}{\partial x_1^2\partial x_2^6}+\frac{\partial^8\psi(x_1,x_2)}{\partial x_2^8}\right)t^4+\frac{a^{10}}{120}\left(\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{10}}+5\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{8}\partial x_2^{2}}+\right.$$

$$\left.+10\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{6}\partial x_2^{4}}+10\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{4}\partial x_2^{6}}+5\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_1^{2}\partial x_2^{8}}+\frac{\partial^{10}\psi(x_1,x_2)}{\partial x_2^{10}}\right)t^5+O(t^6)$$

teorema isbotlandi.

Adabiyotlar

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